How to find a maximum or a minimum

Economics is a study of optimization. Individuals maximize utility. Firms minimize costs. Some optimization methods will be covered here. Optimization problems can be either minimization problems or maximization problems. These problems are often call extreme value problems where an extreme value can refer to either a maximum or a minimum. Somewhat crudely, \( x^* \) is an extreme value (a maximum) of \( f(x) \) if the value of the function at \( x^* \) is larger than at any other point near \( x^* \). Likewise \( x^* \) will be a minimum of \( f(x) \) if the value of the function at \( x^* \) is smaller than at any other point near \( x^* \). It is also customary to call extreme values stationary points.

Using plots

The advent of the computer has made it feasible to use plots as a method of finding stationary points in some cases. Plots do not require any particular mathematical sophistication. Prior to the computer, however, computing the values for the plot and putting the values on the plot would likely be quite laborious.

Example: Find a stationary point of \( f(x) = 10x - 2x^2 \)

\[
\begin{align*}
  \text{f:=} & \ x \rightarrow 10 \cdot x - 2 \cdot x^2 \\
  \text{p1:=plot::Function2d(f(x), x=1..5)} : \\
  \text{plot(p1)}:
\end{align*}
\]
A visual inspection of the plot indicates that the value of $x$ that gives a maximum value of $f(x)$ is approximately $x=2.5$. Next consider finding a stationary point of

- $g:=x\rightarrow-(10\cdot x-2\cdot x^2)$
  
  $x \rightarrow -(10 \cdot x - 2 \cdot x^2)$

- $p1:=\text{plot}::\text{Function2d}(g(x), x=1..5)$
- $\text{plot}(p1)$
In this case the stationary point is one that produces a minimum rather than a maximum. Again the value of the stationary point is approximately $x=2.5$. These two plots suggest the reason for the terminology stationary point. Pretend that the function is a roller coaster and that a ball is on the roller coaster at the highest or lowest point. The ball will stay at rest unless it is disturbed. If the ball is slightly displaced from the stationary point, it will now tend to remain in motion departing from the maximum or oscillating about the minimum.

Plotting can be used to locate stationary points in simple cases. Plots are not very useful if the problem involves more than two independent variables. They can also require significant trial and error. They are not useful if symbolic rather than numerical solutions are needed. They can only produce approximate answers and the approximation may not be particularity accurate. More sophisticated method can be used to produce more accurate answers.

The D operator

MuPAD has a very useful operator called the D operator. An operator is something that performs an operation on a mathematical expression. Mathematicians call the D operator a derivative. Economists call the D operator a marginal change. The D operator computes how much $f(x)$ changes for a given change in $x$. Suppose that the marginal change in $f(x)$ is positive. That means that $f(x)$ will be increasing as $x$ increases. If the marginal change in $f(x)$ is negative, then $f(x)$ will be decreasing as $x$ increases. At a stationary point the function neither increases nor decreases. In other words the marginal change in $f(x)$ will be zero at a stationary point.

This gives a method of locating a stationary point of a function $f(x)$.

1. Apply the D operator to $f(x)$. Call the result $Df$.
2. Find the value of $x$ that makes $Df=0$.

Now let D "operate" on $f(x)$

- $Df:=D(f)$ ;
  
  $x \rightarrow 10 - 4 \cdot x$

- $ans:=solve(Df(x)=0,x)$ ;

  $\begin{cases} 
  5 \\
  2 
  \end{cases}$

- $stationary_x:=op(ans)$ ;

  $\frac{5}{2}$
1. \[ p1:=\text{plot::Function2d}(f(x), \, x=1..5, \, \text{Color=RGB::Red}) ; \]
2. \[ p2:=\text{plot::Function2d}(Df(x), \, x=1..5, \, \text{Color=RGB::Blue}) ; \]
3. \[ p3:=\text{plot::Line2d}([\text{stationary}_x,0],[\text{stationary}_x,f(\text{stationary}_x)], \, \text{Color=RGB::Green}) ; \]
4. \[ \text{plot}(p1,p2,p3, \, \text{AxesTitles=["x","f(x)"]}); \]

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**Figure 3. Locating the stationary point of f(x)**

- \[ Dg:=D(g) ; \]
  \[ x \rightarrow 4 \cdot x - 10 \]
- \[ \text{ans:=solve}(Dg(x)=0,x) ; \]
  \[ \{ \frac{5}{2} \} \]
- \[ \text{stationary}_x:=\text{op}(\text{ans}) ; \]
  \[ \frac{5}{2} \]

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- \[ p1:=\text{plot::Function2d}(g(x), \, x=1..5, \, \text{Color=RGB::Red}) ; \]
- \[ p2:=\text{plot::Function2d}(Dg(x), \, x=1..5, \, \text{Color=RGB::Blue}) ; \]
- \[ p3:=\text{plot::Line2d}([\text{stationary}_x,0],[\text{stationary}_x,g(\text{stationary}_x)], \, \text{Color=RGB::Green}) ; \]
- \[ \text{plot}(p1,p2,p3, \, \text{AxesTitles=["x","g(x)"]}); \]
Actually this scheme locates stationary values of a function (they could be maxima or minima). You can also use the D operator to distinguish between maxima and minima. The following rule can be used to determine whether a stationary point $x^*$ is a maximum or a minimum

1. Apply the D operator twice
2. Evaluate the resulting expression at $x^*$
3. If the value of the resulting expression is
   - negative then the stationary point is a maximum
   - positive then the stationary point is a minimum

Apply the D operator twice to $f(x)$

- delete ans ;
- ans:=x->D(D(f));
  
  \[
  x \rightarrow f''
  \]
- ans(stationary_x);
  
  $-4$

The negative result indicates that the stationary point $x=2.5$ produces a maximum value of $f(x)$.
Next apply the method to \( g(x) \).

- delete \( \text{ans}; \)

- \( \text{ans}:=x\rightarrow D(D(g)) \);
  \[ x \rightarrow g'' \]

- \( \text{ans}(\text{stationary}_x); \)
  \[ 4 \]

The positive result shows that the stationary point \( x=2.5 \) produces a minimum value of \( g(x) \).

**Summary**

To locate a stationary point of a function \( f(x) \)

1. Apply the D operator to \( f(x) \). Call the result \( Df \).
2. Find the value of \( x \) that makes \( Df=0 \).

To determine whether the stationary point is a maximum or a minimum

1. Apply the D operator twice
2. Evaluate the resulting expression at \( x^* \)
3. If the value of the resulting expression is
   - negative then the stationary point is a maximum
   - positive then the stationary point is a minimum